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Solutions of the extended Graetz problem

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Abstract

In this paper laminar forced convective heat transfer problems inside ducts, with axial conduction, subjected to the three main types of boundary conditions are solved exactly. The general method of solution involves a change of the dependent variable leading to a square integrable function in the real line. A complete basis for the vector space of these functions is used to generate an infinite expansion for the solution. The form of solutions is presented for the flows inside a circular pipe, the annular space between pipes, and between parallel plates. © 2000 Elsevier Science Ltd. All rights reserved.

1. Introduction

Convective heat transfer in pipes of simple cross-section has been the subject of a large number of investigations, both of analytical and numerical nature. The original Graetz problem [1] has been extended to cover conditions with low Peclet number, cases in which the axial conduction in the fluid cannot be neglected. In a number of theoretical works [2–13], the flow domain is set as the positive real axis, and the assumption of uniform fluid temperature at the inlet ($z = 0$) is employed. As has been frequently pointed out, when the axial conduction is important, the uniform inlet assumption is invalid and the fluid temperature is altered before the inlet by upstream conduction. Therefore, the domain must be extended, in the limit, requiring, the inlet temperature to be specified at $-\infty$. The literature contains a large number of works using this two-region approach [14–33]. Flows between parallel plates, and in circular ducts have been thoroughly investigated. In all these, the real axis is divided into two regions and, in fact, the problem is treated as two different problems, with different independent variables and discontinuous boundary conditions with a jump at the origin. The two solutions for the divided domain are matched at the origin by the requirement of continuity of temperature and axial heat flux.

A general method of solution of these problems, yielding a single solution valid for the complete domain ($-\infty \leq z \leq \infty$), applicable to the three main types of thermal boundary conditions, to flows between parallel plates, through circular and annular pipes is presented. Hydrodynamic boundary conditions of non-slip at the confining walls are used. Additionally, the thermal boundary conditions are allowed to vary along the axial coordinate in quite a general fashion. The solution is based on the convenient Gram–Charlier basis for the Hilbert space of square

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Nomenclature

A_k, F_k	constants in Eqs. (10) and (11)
a	constant of integration in asymptotic solution (Eq. (6))
Bi	Biot number ($h_s r_0/k$)
b	constant of integration in asymptotic solution (Eq. (6))
f_k	coefficient functions of the expansion in the Gram–Charlier series solution
g_i	function basis of Gram–Charlier
h	convective heat transfer coefficient for the fluid (W/m^2 K)
h_s	convective heat transfer coefficient for the surroundings (W/m^2 K)
h_j	weights defined by $h_j = \mathscr{H}_j(g_j)$
H_j	Hermite polynomial
$\mathscr{H}_j$	operator defined by Eq. (9)
k	fluid thermal conductivity (W/m K)
L	Laplacian operator in one dimension
Nu	Nusselt number (hr_0/k)
Pe	Peclet number ($r_0 v^*/\alpha$)
q	heat flux (W/m^2)
r, r_0, r_E	radial distance, radius of the pipe, radius of external pipe (m)
S	function defined by Eq. (45)
u	arbitrary function of the axial coordinate in Eq. (9)
T	temperature (K)
$v_z(r)$, v^*	fluid velocity profile, reference fluid velocity (m/s)
$v(\eta)$	dimensionless fluid velocity profile, $v = v_z/v^* = v(\eta)$
y	transverse coordinate (m)
z	axial coordinate (m)
Z	dimensionless axial coordinate $Z = z/(r_0 Pe)$

Greek symbols

α	thermal diffusivity (m^2/s)
β, β_∞	asymptotic value of $\partial\theta/\partial Z$
ΔT	reference temperature difference (K)
φ	dimensionless heat flux $q/\lvert q_\infty\rvert$
θ	dimensionless temperature $(T - T_0)/\Delta T$
η	dimensionless transverse coordinate r/r_0 or y/r_0
λ	ratio of outside to inside radii (r_E/r_0)
$\Omega(Z)$	function of axial coordinate specifying a boundary condition (Eq. (48))
ω_k	coefficients for the expansion of $\Omega(Z)$ in (Eq. (48))

Subscripts

0	inlet conditions, at $Z = -\infty$
m	average value
w	values at wall
L	value at lower wall
U	value at upper wall
I	value at internal wall
E	value at external wall
∞	asymptotic value as $Z \to \infty$

integrable functions of a real variable. The construction of the solution involves a change in the dependent variable aiming at its reduction to a square integrable function. For this new variable, a solution is proposed as an expansion in a series with respect to the Gram–Charlier basis. The basis is chosen in view of two important properties. Firstly, each of the basis function is orthogonal to all but one of the Hermite polynomials. Secondly, each vector is generated by the first derivative of the previous one.

$$g_0 = \exp\{-Z^2\}; \quad g_{k+1} = \frac{dg_k}{dz}; \tag{1}$$

$$g_k = (-1)^k H_k(Z) \exp\{-Z^2\}; \tag{2}$$

$$\int_{-\infty}^{\infty} g_i(Z) H_j(Z)\, dZ = \begin{cases} 0 & \text{if } i \neq j \\ h_i = (-1)^i \sqrt{\pi} 2^i i! \end{cases} \tag{3}$$

It must be stressed that this function basis starts with $k = 0$, as there are no Hermite polynomial of negative orders. This allows the establishment of an infinite set of ordinary differential equations for the coefficients of the solution expansion. Each equation depends exclusively on the two previous ones, a fact that allows their solution in sequence, with no recourse to approximations of any kind.

2. Problem formulation

Consider the fully developed flow of a Newtonian fluid in ducts consisting of infinitely long, either: parallel plates, cylindrical pipe, or the annular region between cylindrical pipes. Viscous energy dissipation and other forms of heat generation are neglected. The general configuration is presented in Fig. 1. The energy balance between convection and conduction of heat in the fluid gives:

$$L\theta + \frac{1}{Pe^2}\frac{\partial^2\theta}{\partial Z^2} = v(\eta)\frac{\partial\theta}{\partial Z}, \tag{4}$$

where:

$$L\theta = \frac{1}{\eta}\frac{\partial}{\partial\eta}\left[\eta\frac{\partial\theta}{\partial\eta}\right], \text{ or } L\theta = \frac{\partial^2\theta}{\partial\eta^2}, \quad \text{and} \quad \eta = \frac{r}{r_0} \text{ or}$$

$$\eta = \frac{y}{r_0}; \quad Z = \frac{z}{r_0 Pe}; \quad Pe = \frac{r_0 v^*}{\alpha}, \quad v(\eta) = \frac{v_z}{v^*}, \tag{5}$$

$$\text{and} \quad \theta = \frac{T - T_0}{\Delta T}.$$

In the above equations, T_0 is a reference temperature and ΔT is reference temperature difference. In all cases it will be assumed that the limit temperature ($\lim_{Z\to-\infty} T(Z, \eta) = T_0$) exists, and convergence is fast enough to guarantee the square-integrability. The proposed method can be applied to the three main types of convective heat transfer problems: (1) prescribed wall temperatures; (2) prescribed wall heat flux; (3) prescribed convective heat transfer to the surroundings. Furthermore, the specified wall conditions are allowed to vary with the axial coordinate. The velocity profile is assumed to be a function of the transverse coordinate only, satisfying the non-slip condition at the confining walls.

The present method of solution depends upon the determination of an asymptotic solution, valid for large values of Z. This arises from a simplified form of the energy balance (4), in which the first derivative of θ with respect to Z is substituted by its asymptotic constant value, which is assumed to exist, and consequently, the second derivative is set to zero. In this case, Eq. (4) becomes: $L\theta_\infty = \beta_\infty v(\eta)$, where $\beta_\infty = \lim_{Z\to\infty}\partial\theta/\partial Z$. Notice that the right-hand side of this equation is a function of η only, and thus it yields a solution independent of Z. In a procedure analogous to the "variation of parameters", all the parameters of the solution, constants of integration and β (replacing β_∞), are allowed to vary with Z. The existence of the limit β_∞ is crucial. For the cases of specified wall temperature and convective boundary conditions this limit is zero ($\beta_\infty = 0$). For the case of specified heat flux, β_∞ is a known constant, related to the limiting value of the heat flux. Therefore, the three main kinds of heat transfer problems can be solved. Some cases of mixed boundary conditions can also be solved. The proposed solution is:

$$\theta = \theta_\infty + \sum_{k=0}^{\infty} f_k(\eta) g_k(Z) \quad \text{where}$$
$$\theta_\infty = \beta(Z)L^- v(\eta) + a(Z)L^-(0) + b(Z). \tag{6}$$

In this expression L^- is the pseudo-inverse of the operator, calculated by the expressions:

$$L^- v = \begin{cases} \displaystyle\int \frac{1}{\eta}\int \eta v(\eta)\,d\eta\,d\eta \\ \displaystyle\int\int v(\eta)\,d\eta\,d\eta \end{cases}, \quad \text{and} \quad L^-(0) = \begin{cases} \ln\eta \\ \eta \end{cases}, \tag{7}$$

respectively, for cylindrical or Cartesian coordinates. Let the parameters of the asymptotic solution depend on Z allows the functions in the asymptotic solution to be specified, in such a way as to satisfy the inlet temperature: $\lim_{Z\to-\infty}\theta_\infty(Z,\eta) = 0$. This requires that

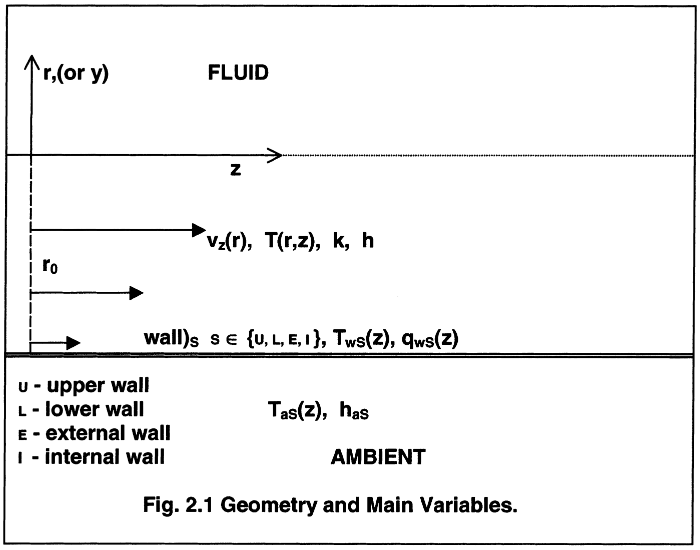

Fig. 1. Geometry and main variables.

$\lim_{Z\to-\infty}\beta(Z) = \lim_{Z\to-\infty}a(Z) = \lim_{Z\to-\infty}b(Z) = 0$. The corresponding limits for large values of Z must exist since θ_∞ must reproduce the original asymptotic solution. Therefore, θ and θ_∞ both vanish at the left limit ($Z \to -\infty$). Since θ approaches θ_∞ as Z increases, so $\theta - \theta_\infty$ approaches zero at both ends of the real line. It is assumed that convergence is fast enough to render the difference $\theta - \theta_\infty$ a square integrable function. This argument allows its expansion in terms of the Gram–Charlier series.

Substitution of the proposed solution (Eq. (6)) into Eq. (4), yields the following intermediate result:

$$\sum_{k=0}^{\infty}\left(Lf_k - vf_{k-1} + \frac{1}{Pe^2}f_{k-2}\right)g_k$$
$$= (b' - \beta)v + \left(\beta' v - \frac{\beta''}{Pe^2}\right)L^{-}v + \left(a'v - \frac{a''}{Pe^2}\right)L^{-}(0) - \frac{b''}{Pe^2}, \tag{8}$$

in which the primes denote differentiation with respect to Z.

The left-hand side of this equation approach zero as $Z \to \infty$, implying the same to be true for the right side. The first term on the right-hand side contains β, which in problems of the second kind, is assumed to approach a limiting value different from zero, requiring, in such cases, that b' be set equal to β ($b' = \beta$). In all the other instances, $\beta = 0$, and b' and all the remaining derivatives, converge to zero.

The application of the following operators defined for an arbitrary function of $u(Z)$:

$$\mathscr{H}_j(u) = \frac{1}{h_j}\int_{-\infty}^{\infty} uH_j(Z)\,\mathrm{d}Z, \tag{9}$$

lead to the following important result:

$$Lf_j = v(A_j + f_{j-1}) - \frac{B_j + f_{j-2}}{Pe^2} + \left(C_j v - \frac{D_j}{Pe^2}\right)L^{-}v + \left(E_j v - \frac{F_j}{Pe^2}\right)L^{-}(0), \tag{10}$$
$$f_{-1} = f_{-2} = 0$$

where:

$$A_j = \mathscr{H}_j(b' - \beta), \quad C_j = \mathscr{H}_j(\beta'), \quad E_j = \mathscr{H}_j(a'),$$

$$B_j = \mathscr{H}_j(b''), \quad D_j = \mathscr{H}_j(\beta''), \quad F_j = \mathscr{H}_j(a''). \tag{11}$$

The boundary conditions determine the parameters β, a and b as functions of Z, allowing the determination of the constants A–F, in Eq. (10). Notice that f_0 depends only on A_0–F_0; f_1 depends on A_1–F_1 and on f_0; and, in general, f_j depends on A_j–F_j, and on f_{j-1}, f_{j-2}. The infinite set of ordinary linear differential equations can be sequentially solved, exactly. For the parabolic velocity profiles, the solution yields polynomials in η of increasing degrees. The flow in annular regions introduces terms containing logarithms in the solution. At this point, it is added that the method can be applied to flows of non-Newtonian fluids pending the absence of secondary flows, and the existence of an analytical expression for the integral $L^{-}v$. Power law fluids introduce non-integer exponents. These cases can be treated equally well, and in fact, more complex models for non-Newtonian fluids can be used, as long as an analytical expression for the velocity profile exists.

The steps reported below should be followed:

(a) use the boundary conditions to determine the functions $a(Z)$, $b(Z)$ and $\beta(Z)$,
(b) determine $L^{-}v$ by the integration of the velocity profile (required only if $\beta \neq 0$),
(c) determine the constants A–F in Eq. (10), with the use of Eq. (11),
(d) determine, sequentially, the coefficient functions f_k.

Some cases will be examined in detail.

3. Specified wall temperature

3.1. Circular pipes

In the simplest case of flows inside circular pipes the wall temperature is given as a function of the axial coordinate satisfying, by hypothesis, the two limits given below

$$\lim_{z\to-\infty} T_\mathrm{w}(z) = T_0, \quad \text{and} \quad \lim_{z\to+\infty} T_\mathrm{w}(z) = T_{\mathrm{w}\infty} \tag{12}$$

Let $\Delta T = T_{\mathrm{w}\infty} - T_0$, with which the dimensionless temperatures become:

$$\theta(\eta, Z) = \frac{T - T_0}{T_{\mathrm{w}\infty} - T_0}, \quad \text{and} \quad \theta_\mathrm{w}(Z) = \frac{T_\mathrm{w} - T_0}{T_{\mathrm{w}\infty} - T_0} \tag{13}$$

The dimensionless wall temperature is assumed to be a continuous function of Z almost everywhere, ranging from zero at the left limit, and increases monotonically to the right limit. Notice that the discontinuous condition, $\theta_\mathrm{w} = 0$ for $Z < 0$, and $\theta_\mathrm{w} = 1$ for $Z \geq 0$, satisfy these assumptions.

The temperature difference between the fluid and the wall is small everywhere, $|\theta - \theta_\mathrm{w}| \ll 1$. In consequence of this, $(\theta - \theta_\mathrm{w})^2$ is even more smaller, and approaches zero for large values of $\|Z\|$. This difference decreases with increasing thermal conductivity, and decreasing fluid velocity; the smaller is the Peclet number the smaller will be the square of the temperature difference. In fact, the temperature difference is small for all values of the Peclet number, and the solution reproduces the results obtained in the limit as $Pe \to \infty$. This argument justifies the assumption of square integrability; i.e. the difference in fluid to wall temperature belongs to the Hilbert space of square integrable functions of a real variable. Notice that in this case β is zero, and the boundary conditions imply $a = 0$ and $b = \theta_w$. The asymptotic solution is $\theta_\infty = \theta_w$. Thus, the proposed solution is of the form:

$$\theta(Z, \eta) = \theta_w(Z) + \sum f_k(\eta) g_k(Z). \tag{14}$$

The boundary conditions for θ are:

$$\frac{\partial \theta}{\partial \eta}(Z, 0) = 0 \twoheadrightarrow f'_k(0) = 0, \quad \text{and} \qquad \theta(Z, 1) = \theta_w(Z) \twoheadrightarrow f_k(1) = 0. \tag{15}$$

In view of these results, it is established that only A_j and B_j differ from zero, and Eq. (10) reduces to:

$$Lf_j = v(A_j + f_{j-1}) - \frac{1}{Pe^2}(B_j + f_{j-2}) \tag{16}$$

where:

$$A_j = \mathscr{H}_j(\theta'_w) = \frac{1}{h_j}\int_{-\infty}^{+\infty} \theta'_w(Z) H_j(Z)\,dZ, \quad \text{and} \qquad B_j = \mathscr{H}_j(\theta''_w) = \frac{1}{h_j}\int_{-\infty}^{+\infty} \theta''_w(Z) H_j(Z)\,dZ. \tag{17}$$

Furthermore, $f_{-1} = f_{-2} = 0$. The boundary conditions for these functions arise from Eq. (15). The numbers A_j and B_j are determined with the knowledge of the wall temperature. The first coefficient function f_0 depends only on A_0 and B_0; f_1 depends on A_1, B_1 and f_0; the generic term f_j depends on A_j, B_j, f_{j-1}, and f_{j-2}. They can be sequentially calculated exactly, no approximate procedure is required. The solutions of Eq. (16) are polynomials of degree $4j + 4$. A simple Maple program was established to perform the integrations. Detailed results are presented in Ref. [34].

3.2. Parallel plates

For the flow between parallel plates separated by a distance of $2r_0$, two wall temperature distributions must be specified. It is assumed that the upper and lower walls, and the fluid, have the same inlet temperature ($\lim_{z\to-\infty} T(z, y) = \lim_{z\to-\infty} T_U = \lim_{z\to-\infty} T_L = T_0$). Both walls are assumed to have varying temperatures, which attain asymptotic, possibly unequal ($T_{U\infty}$, and $T_{L\infty}$) values. The dimensionless temperatures are defined as:

$$\theta = \frac{T - T_0}{T_{L\infty} - T_0}, \quad \theta_L = \frac{T_L - T_0}{T_{L\infty} - T_0}, \quad \text{and} \qquad \theta_U = \frac{T_U - T_0}{T_{L\infty} - T_0}. \tag{18}$$

The proposed solution is:

$$\theta = \theta_\infty + \sum f_k g_k, \qquad \theta_\infty = \frac{1}{2}(\theta_U - \theta_L)\eta + \frac{1}{2}(\theta_U + \theta_L). \tag{19}$$

The leading terms grouped as θ_∞ reproduce the temperatures of the two walls ($\eta = 1$, and $\eta = -1$). They represent the asymptotic solution, for large values of Z, of the energy equation in the form $L\theta_\infty = 0$, with boundary conditions depending on Z. The left limit for θ is zero, and the right limit is the asymptotic solution. Therefore, $\theta - \theta_\infty$ approaches zero on both sides of the real line. The comparison between Eqs. (19) and (6) shows that $a = \frac{1}{2}(\theta_U - \theta_L)$, $b = \frac{1}{2}(\theta_U - \theta_L)$ and that $\beta = 0$. Eq. (10) becomes:

$$Lf_j = v(A_j + f_{j-1}) - \frac{B_j + f_{j-2}}{Pe^2} + \left(E_j v - \frac{F_j}{Pe^2}\right)\eta, \tag{20}$$

where

$$A_j = \frac{1}{2}\mathscr{H}_j[\theta'_U + \theta'_L], \quad E_j = \frac{1}{2}\mathscr{H}_j[\theta'_U - \theta'_L],$$

$$B_j = \frac{1}{2}\mathscr{H}_j[\theta''_U + \theta''_L], \quad F_j = \frac{1}{2}\mathscr{H}_j[\theta''_U - \theta''_L]. \tag{21}$$

The boundary conditions are readily determined to be homogeneous since the asymptotic part of the solution reproduces the two wall temperatures; $f_j(-1) = f_j(1) = 0$. All comments about the method and form of the solution given in the previous section are equally valid.

3.3. Annular region

In principle, this is the exact analog of the previous problem where the internal T_I, and external T_E, wall temperatures are specified in the limits of the domain $r_0 \le r \le \lambda r_0$, i.e. $1 \le \eta \le \lambda$, and the dimensionless temperatures are defined as:

$$\theta = \frac{T - T_0}{T_{I\infty} - T}, \quad \theta_I = \frac{T_I - T_0}{T_{I\infty} - T_0}, \quad \text{and} \qquad \theta_E = \frac{T_E - T_0}{T_{I\infty} - T_0}. \tag{22}$$

Hence:

$$\theta(Z,1)=\theta_{\mathrm{I}}(Z),\quad \text{and}\quad \theta(Z,\lambda)=\theta_{\mathrm{E}}(Z). \tag{23}$$

The proposed solution contains an asymptotic term determined as a solution of $L\theta_\infty=0$. It differs from the previous case only in consequence of the form of the laplacian operator in cylindrical coordinates.

$$\theta=\theta_\infty+\sum f_k g_k,\quad \theta_\infty=(\theta_{\mathrm{E}}-\theta_{\mathrm{I}})\frac{\ln\eta}{\ln\lambda}+\theta_{\mathrm{I}}. \tag{24}$$

By the arguments presented in the previous case, one obtains homogeneous boundary conditions, and an infinite set of ordinary differential equations to be satisfied by the coefficient functions, they are:

$$f_j(1)=f_j(\lambda)=0. \tag{25}$$

Comparison of Eqs. (24) and (6) permits the identification of the functions $\beta=0$, $a=(\theta_{\mathrm{E}}-\theta_{\mathrm{I}})/\ln\lambda$ and $b=\theta_{\mathrm{I}}$ and the calculation of the constants in Eq. (10).

$$\begin{aligned}Lf_j&=v\big(A_j+f_{j-1}\big)-\frac{B_j+f_{j-2}}{Pe^2}\\&\quad+\left(E_jv-\frac{F_j}{Pe^2}\right)\ln\eta;\end{aligned} \tag{26}$$

$$A_j=\mathscr{H}_j\big[\theta_I'\big],\quad E_j=\frac{1}{\ln\lambda}\mathscr{H}_j\big[\theta_{\mathrm{E}}'-\theta_{\mathrm{I}}'\big],$$

$$B_j=\mathscr{H}_j\big[\theta_{\mathrm{I}}''\big],\quad F_j=\frac{1}{\ln\lambda}\mathscr{H}_j\big[\theta_{\mathrm{E}}''-\theta_{\mathrm{I}}''\big]. \tag{27}$$

4. Specified heat flux

It is assumed, as previously, that the fluid entering at temperature T_0, and that the heat flux at the wall is specified as a function of z satisfying the following limits: $\lim_{z\to-\infty}q_{\mathrm{w}}(z)=0$, and $\lim_{z\to\infty}q_{\mathrm{w}}(z)=q_\infty$. If two walls are present, then the heat flux is specified at both, the left limit must apply to both, but the right limits may differ in sign or in absolute value. The simplest problem is again the one in circular pipes.

4.1. Circular pipes

The dimensionless energy balance equation is invariant, and only the reference ΔT is defined differently as $\Delta T=|q_\infty|r_0/k$. The boundary conditions are:

$$\lim_{Z\to-\infty}\theta(Z,\eta)=0,\quad \frac{\partial T}{\partial r}(z,0)=0\twoheadrightarrow\frac{\partial\theta}{\partial\eta}(Z,0)=0,$$

$$k\frac{\partial T}{\partial r}(z,r_0)=q_{\mathrm{w}}(z)\twoheadrightarrow\frac{\partial\theta}{\partial\eta}(Z,1)=\frac{q_{\mathrm{w}}(z)}{|q_\infty|}=\varphi(Z). \tag{28}$$

The proposed solution is of the general form of the asymptotic solution to which the expansion in the Gram–Charlier series is added.

$$\begin{aligned}&\theta=\theta_\infty+\sum f_k g_k,\\&\theta_\infty=4\varphi(Z)L^-v+4\int_{-\infty}^{Z}\varphi(Z)\,\mathrm{d}Z.\end{aligned} \tag{29}$$

The pseudo inverse obtained from Eq. (7) is given by $L^-v=\frac{1}{4}(1-\frac{1}{2}\eta^2)\eta^2$. The term in the integral corresponds to term $4Z$ that appears in the asymptotic solution for the constant heat flux problem, and without axial conduction. As already mentioned $b'=\beta$, which implies $A_j=0$.

As the asymptotic part of the proposed solution satisfies the boundary conditions, then the coefficient function of the expansion satisfies homogeneous conditions.

$$Lf_j=vf_{j-1}-\frac{B_j+f_{j-2}}{Pe^2}+\left(C_jv-\frac{D_j}{Pe^2}\right)L^-v. \tag{30}$$

$$B_j=C_j=4\mathscr{H}_j\big(\varphi'\big);\qquad D_j=4\mathscr{H}_j\big(\varphi''\big). \tag{31}$$

Knowledge of the heat flux distribution allows the calculation of B_j, C_j and D_j for all values of j. As in the previous situations the equation for f_0 depends only on B_0, C_0 and D_0, the equation for f_1 depends only on B_1, C_1, D_1 and f_0, and the general term f_j depends only on B_j, C_j, D_j and f_{j-1}, f_{j-2}. Details of this solution will be presented elsewhere.

4.2. Parallel plates

The parabolic velocity profile yields $L^-v=\frac{1}{2}(1-\frac{1}{6}\eta^2)\eta^2$ and $L^-(0)=\eta$. The general form of the solution is given by Eq. (29) with a new definition for the asymptotic part:

$$\begin{aligned}\theta_\infty&=\frac{3}{4}(\varphi_{\mathrm{U}}-\varphi_{\mathrm{L}})L^-v+\frac{1}{2}(\varphi_{\mathrm{U}}+\varphi_{\mathrm{L}})\eta\\&\quad+\frac{3}{4}\int_{-\infty}^{Z}(\varphi_{\mathrm{U}}-\varphi_{\mathrm{L}})\,\mathrm{d}Z,\end{aligned} \tag{32}$$

where φ_{U} and φ_{L} are the dimensionless heat fluxes, respectively, for $\eta=1$, and $\eta=-1$. The constants in Eq. (10) are:

$$B_j=C_j=\frac{3}{4}\mathscr{H}_j\big[\varphi_{\mathrm{U}}'-\varphi_{\mathrm{L}}'\big],\quad D_j=\frac{3}{4}\mathscr{H}_j\big[\varphi_{\mathrm{U}}''-\varphi_{\mathrm{L}}''\big],$$

$$E_j = \frac{1}{2}\mathscr{H}_j[\varphi_U' + \varphi_L'], \quad \text{and}$$
$$F_j = \frac{1}{2}\mathscr{H}_j[\varphi_U'' + \varphi_L'']. \tag{33}$$

Knowledge of the heat flux on both walls is necessary, and sufficient to determine the above constants. Comments on the structure of the set of differential equations given in the previous section also apply here.

4.3. Annular region

The steps for the determination of the asymptotic solution follow, very closely those of the previous item. Firstly L^-v, and $L^-(0)$ are determined; from these the general expression for θ_∞ follows. The boundary conditions for the heat flux in the internal and external walls are used to determine the functions $\beta(Z)$, and $a(Z)$ appearing in the general solution, Eq. (6). Repeating the arguments given above there follows $b' = \beta$.

$$\begin{aligned}\theta_\infty &= \beta L^- v + a \ln \eta + b \\ &= \beta\left[\frac{1}{4}\eta^2\left(1 - \frac{1}{4}\eta^2\right) + \frac{\lambda^2 - 1}{4 \ln \lambda}\eta^2(\ln \eta - 1)\right] \\ &\quad + a \ln \eta + b,\end{aligned} \tag{34}$$

The boundary conditions at each wall are written as:

$$-\frac{\partial \theta}{\partial \eta}(Z, 1) = -\beta S^{\mathrm{I}}(\lambda) + a = \varphi^{\mathrm{I}}$$

$$-\frac{\partial \theta}{\partial \eta}(Z, \lambda) = -\beta S_{\mathrm{E}}(\lambda) + \frac{a}{\lambda} = \varphi_{\mathrm{E}},$$

where

$$S_{\mathrm{I}}(\lambda) = \left[\frac{1}{4} - \frac{(\lambda^2 - 1)}{4 \ln \lambda}\right],$$
$$S_{\mathrm{E}}(\lambda) = \left[\frac{\lambda^3}{4} + \frac{\lambda(1 - \lambda^2)}{4 \ln \lambda}\right] \tag{35}$$

Introducing these results into the general asymptotic solution (Eq. (34)) gives:

$$\begin{aligned}\theta &= \theta_\infty + \sum f_k g_k, \\ \theta_\infty &= \frac{\lambda\varphi_{\mathrm{E}} - \varphi_{\mathrm{I}}}{\lambda S_{\mathrm{E}} - S_{\mathrm{I}}} L^- v \\ &\quad + \left(\varphi_{\mathrm{I}} - \frac{\lambda\varphi_{\mathrm{E}} - \varphi_{\mathrm{I}}}{\lambda S_{\mathrm{E}} - S_{\mathrm{I}}} S_{\mathrm{I}}\right)\ln \eta + \int_{-\infty}^{Z} \frac{\lambda\varphi_{\mathrm{E}} - \varphi_{\mathrm{I}}}{\lambda S_{\mathrm{E}} - S_{\mathrm{I}}}\,\mathrm{d}Z.\end{aligned} \tag{36}$$

Eq. (10) is applicable with the constants calculated from the expression (11):

$$B_j = C_j = \frac{1}{\lambda S_{\mathrm{E}} - S_{\mathrm{I}}}\mathscr{H}_j(\lambda\varphi_{\mathrm{E}}' - \varphi_{\mathrm{I}}'),$$
$$D_j = \frac{1}{\lambda S_{\mathrm{E}} - S_{\mathrm{I}}}\mathscr{H}_j(\lambda\varphi_{\mathrm{E}}'' - \varphi_{\mathrm{I}}''),$$

$$E_j = \mathscr{H}_j\left(\varphi_{\mathrm{I}}' - \frac{\lambda\varphi_{\mathrm{E}}' - \varphi_{\mathrm{I}}'}{\lambda S_{\mathrm{E}} - S_{\mathrm{I}}}S_{\mathrm{I}}\right),$$
$$F_j = \mathscr{H}_j\left(\varphi_{\mathrm{I}}'' - \frac{\lambda\varphi_{\mathrm{E}}'' - \varphi_{\mathrm{I}}''}{\lambda S_{\mathrm{E}} - S_{\mathrm{I}}}S_{\mathrm{I}}\right) \tag{37}$$

5. Specified convective transfer

The third type of boundary conditions deal with problems of convective heat transfer from the wall surfaces to the surroundings, neglecting the wall thermal resistance. The fluid inlet temperature is T_0, and the surroundings temperature may vary from T_0, for all surfaces, to an asymptotic value $T_{\mathrm{a}\infty}$, for each surface, with one of these taken as reference. In terms of dimensionless variables, the convective boundary conditions may be written as:

$$\left(\frac{\partial \theta}{\partial \eta}\right)_{\mathrm{w}} = Bi(\theta_{\mathrm{a}} - \theta_{\mathrm{w}}), \quad \text{where } \theta = \frac{T - T_0}{T_{\mathrm{a}\infty} - T_0},$$
$$\theta_{\mathrm{w}} = \frac{T_{\mathrm{w}} - T_0}{T_{\mathrm{a}\infty} - T_0},\ \theta_{\mathrm{a}} = \frac{T_{\mathrm{a}} - T_0}{T_{\mathrm{a}\infty} - T_0}. \tag{38}$$

The Biot number is defined by $Bi = h_{\mathrm{s}}r_0/k$, where h_{s} is the convective heat transfer coefficient from the wall to the surroundings. In this problem $\beta = 0$. The form of the differential equation for the coefficient functions are equal to the equivalent Eqs. (16) and (17) for pipe flow, (19) and (20) for parallel plates, and (26) and (27) for annular flow. For the asymptotic solution one must express the wall temperatures as functions of the specified ambient temperatures.

The coefficient functions satisfy the convective boundary conditions at all walls, expressed as:

$$f_k(1) + Bi\, f_k'(1) = 0. \tag{39}$$

5.1. Circular pipes

The asymptotic solution is determined solely by the surroundings temperature, $\theta_\infty = \theta_{\mathrm{a}}$ ($\beta = 0$, $a = 0$, and $b = \theta_{\mathrm{a}}$). Only the coefficients A_j and B_j differ from zero and the differential equations for the coefficient functions are given by Eq. (16), in which:

$$A_j = \mathscr{H}_j(\theta'_a), \quad \text{and} \quad B_j = \mathscr{H}_j(\theta''_a). \tag{40}$$

An article containing details of this solution has been submitted for publication [35].

5.2. Parallel plates

The asymptotic solution for this case depends on the surroundings temperatures prevailing on the upper and lower sides of the two walls. Let θ_{aU}, and θ_{aL} be the dimensionless ambient temperatures. The asymptotic solution is given by:

$$\theta_\infty = \frac{1}{2}(\theta_{wU} - \theta_{wL})\eta + \frac{1}{2}(\theta_{wU} + \theta_{wL})$$

$$= \frac{\theta_{aU} - \theta_{aL}}{(2 + 1/Bi_L - 1/Bi_U)}\eta + \frac{\theta_{aL}(1 + 1/Bi_U) + \theta_{aU}(1 + 1/Bi_L)}{(2 + 1/Bi_U + 1/Bi_L)} \tag{41}$$

The differential equations for the coefficient functions are given by the set (19), where the constants are determined by expressions similar to (20).

$$A_j = \frac{1}{2}\mathscr{H}_j[\theta'_{wU} + \theta'_{wL}], \quad E_j = \frac{1}{2}\mathscr{H}_j[\theta'_{wU} - \theta'_{wL}],$$

$$B_j = \frac{1}{2}\mathscr{H}_j[\theta''_{wU} + \theta''_{wL}], \quad F_j = \frac{1}{2}\mathscr{H}_j[\theta''_{wU} - \theta''_{wL}] \tag{42}$$

The wall temperatures can be read from Eq. (41).

5.3. Annular region

Surrounding temperature inside the internal wall, and outside the external wall should be prescribed, leading to the dimensionless temperatures θ_{aI} and θ_{aE}. The asymptotic solution is

$$\theta_\infty = (\theta_{wE} - \theta_{wI})\frac{\ln\eta}{\ln\lambda} + \theta_{wI}$$

$$= \frac{\theta_{aE} - \theta_{aI}}{1 + 1/Bi_E + 1/Bi_I}\frac{\ln\eta}{\ln\lambda} + \frac{\theta_{aI}(\ln\lambda + 1/Bi_E) + \theta_{aE}/Bi_I}{\ln\lambda + 1/Bi_E + 1/Bi_I}, \tag{43}$$

on the basis which the constants in Eq. (10) are calculated

$$A_j = \mathscr{H}_j(\theta'_{wI}), \quad E_j = \frac{1}{\ln\lambda}\mathscr{H}_j(\theta'_{wE} - \theta'_{wI}),$$

$$B_j = \mathscr{H}_j(\theta''_{wI}), \quad F_j = \frac{1}{\ln\lambda}\mathscr{H}_j(\theta''_{wE} - \theta''_{wI}). \tag{44}$$

6. Specified heat flux in a circular pipe

In order to demonstrate the *possibilities* of the present method, it is applied to the case of heat transfer to the flow of fluid inside a circular pipe with specified, variable surface heat flux.

Consider the hydrodynamically developed flow in a pipe, with thermal boundary condition of the second type. The dimensionless variables and the boundary conditions are given in Eq. (28), the general form of the solution satisfies Eq. (29), the differential equations for the coefficient functions are given by Eqs. (30) and (31). The problem is completely determined with the specification of the analytic form of the dimensionless heat flux distribution along the Z-axis. This is chosen as a one-parameter (S) family of distributions which approach the step function as $S \to \infty$.

$$\frac{q_w(z)}{|q_\infty|} = \varphi(Z) = \frac{1}{2}\left[1 + \mathrm{erf}(SZ)\right]. \tag{45}$$

The asymptotic solution and the constants appearing in the differential equation for the coefficient functions become known, their values being listed in Table 1 for two values of the parameter S ($S = 1$, and 2). For $S = 1$, the first derivative of the wall heat flux is equal to g_0, and therefore, only A_0, B_0, and C_1 are different from zero. For $S = 2$, and in fact for all other values of S, the first derivative of the wall heat flux is an even function of Z, and consequently, only the B_j and C_j for even, and D_j for odd values of j, differ from zero. Solutions of the set of equations (30), with homogeneous boundary conditions $f'_k(0) = f'_k(1) = 0$ give origin to polynomials in η, of degree $2k + 8$, calculated using a Maple program. The temperature profiles, as a function of η, for different value of the axial coordinate, are given in Fig. 2(a) and (b), respectively, for $S = 1$, and $S = 2$. The heating process leads temperature profiles decreasing towards the center line. Heating starts before the origin of the axial direction (negative values of Z in Fig. 2(a) and (b)) firstly because of the shape of the heating curve, and sec-

Table 1
Coefficients of the differential equations

	$B_i = C_i = D_{i+1}$	
	$S = 1$	$S = 2$
0	2.2568	2.2568
1	0	0
2	0	−0.4231
3	0	0
4	0	0.3967
5	0	0
6	0	−0.0025

ondly due to the axial conduction. The first effect is attenuated with increasing values of S, leading to lower temperatures for the same values of Z. As Z increases, the dimensionless heat flux approaches one, and the temperature profiles become parallel displacements of each other. Qualitative agreement with the results established by Papoutsakis et al. (1980), is evident. Complete agreement cannot be sought due to the difference in boundary conditions.

The results obtained allow the calculation of the wall, and average dimensionless temperatures. Knowledge of these, and of the heat flux as given by the boundary condition (Eq. (45)) allows the calculation of the Nusselt number.

$$Nu = \frac{hr_0}{k} = \frac{q_w(Z)}{T_w - T_m}\frac{r_0}{k} = \frac{\varphi(Z)}{\theta_w(Z) - \theta_m(Z)}, \tag{46}$$

where:

$$\theta_m = 2\int_0^1 v(\eta, Z)\, d\eta. \tag{47}$$

Calculated values are given in Figs. 3(a) and (b) for different values of the Peclet number, and the two values of S. It is observed that the Nusselt number attains an asymptotic value as $Z \to \infty$ independent of the Peclet number and S. This asymptotic value is 4.3636 which compares well with the calculated value

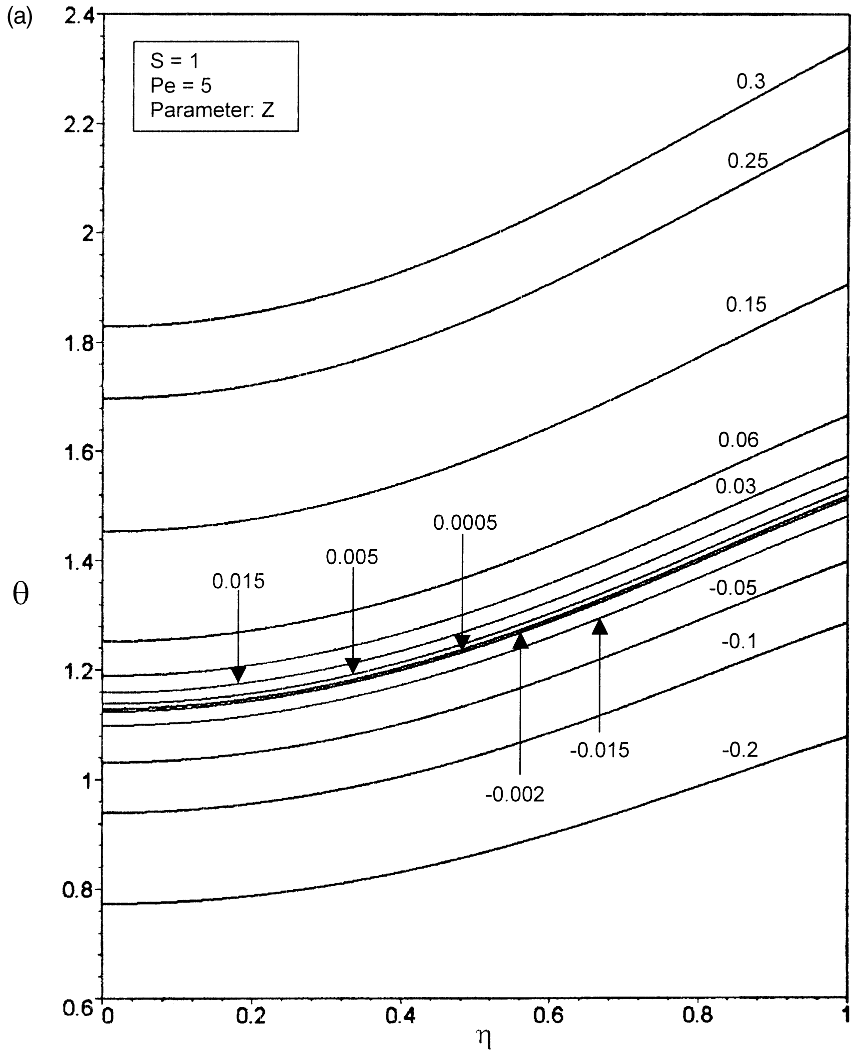

Fig. 2. Radial profiles of dimensionless temperature of various axial distances: (a) $Pe = 5$, $S = 1$; (b) $Pe = 5$, $S = 2$

of 4.36 given by Papoutsakis et al. (1980). It must be stressed that the numerical results for $S = 1$ were calculated with only five terms of the series, as the number of terms required for convergence in four decimal digits. $S = 2$ requires 12 terms, and different problems require even more terms.

7. General representation of the boundary conditions

The functions specifying the boundary conditions must possess asymptotes on both sides of the real line. The left limit being zero, and the right a known constant, say ω_∞. These boundary conditions can be transformed into a square integrable function by subtracting of a term proportional to the error function and the residue can be expanded in a Gram–Charlier series.

$$\Omega(Z) = \frac{\omega_\infty}{2}\left[1 + \mathrm{erf}(Z)\right] + \sum \omega_k g_k. \tag{48}$$

In the above expression, $\Omega(Z)$ stands for any one of the functions of the axial coordinate which specify the boundary condition of the three types considered. The constant ω_∞ is the asymptotic value of $\Omega(Z)$, $\omega_\infty = \lim_{Z\to\infty}\Omega(Z)$, and the coefficients ω_k are calculated with the help of the operators defined by Eq. (9).

$$\omega_k = \mathscr{H}_k\left[\Omega(Z) - \frac{\omega_\infty}{2}(1 + \mathrm{erf}(Z))\right]. \tag{49}$$

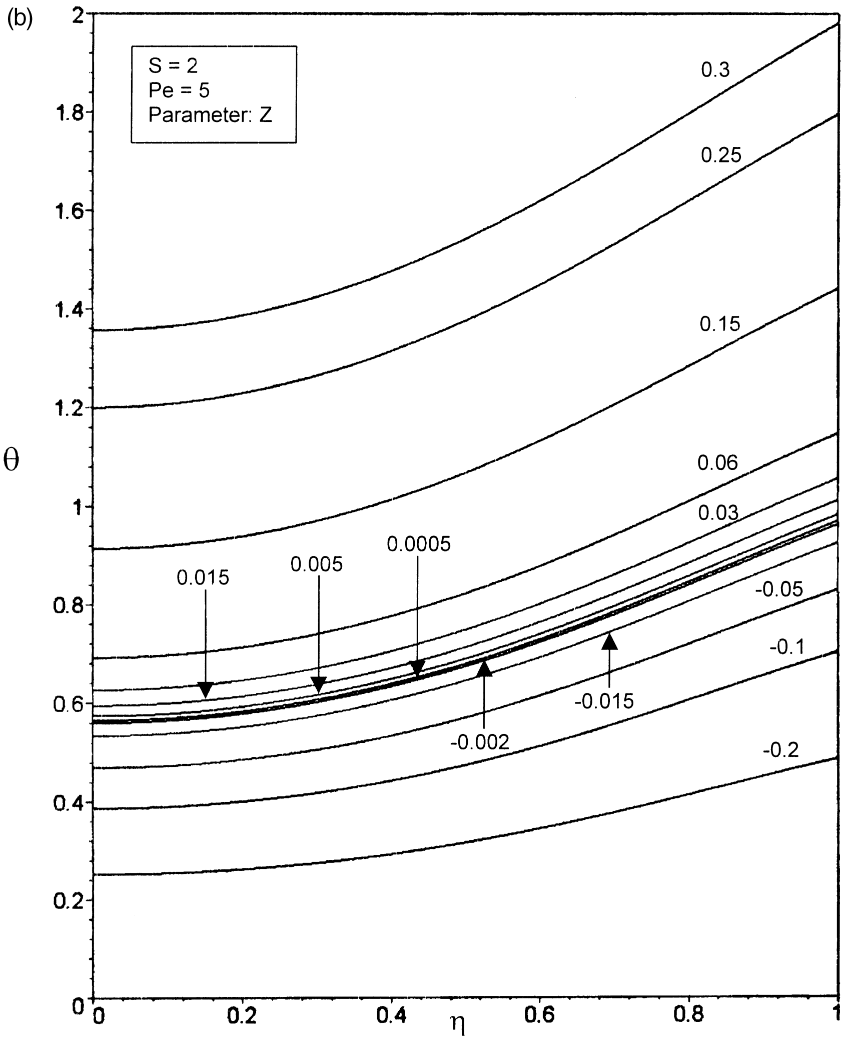

Fig. 2 (*continued*)

Define the set of solutions of the equations for the coefficient functions $\{f_j^{\text{erf}}, f_{j,0}, f_{j,1}, f_{j,2}, \ldots\}$ corresponding to the following set of boundary conditions: $\{\frac{1}{2}[1+\text{erf}(Z)],\ g_0(Z),\ g_1(Z),\ g_2(Z), \ldots\}$. Then the superposition principle implies that the general solution corresponding to the boundary condition (47) is given by the linear combination of the basic solutions defined above. That is:

$$f_j = \omega_\infty f_j^{\text{erf}} + \sum_{i=0} \omega_i f_{j,i}. \tag{50}$$

The proof of this statement rests on the linearity of the operator $\mathscr{H}_j$ with which the constants in the differential equations for the functions f_j are calculated. If A_j^{erf}, and $A_{j,i}$ are the constants corresponding to a boundary condition in the error function, and g_i, then

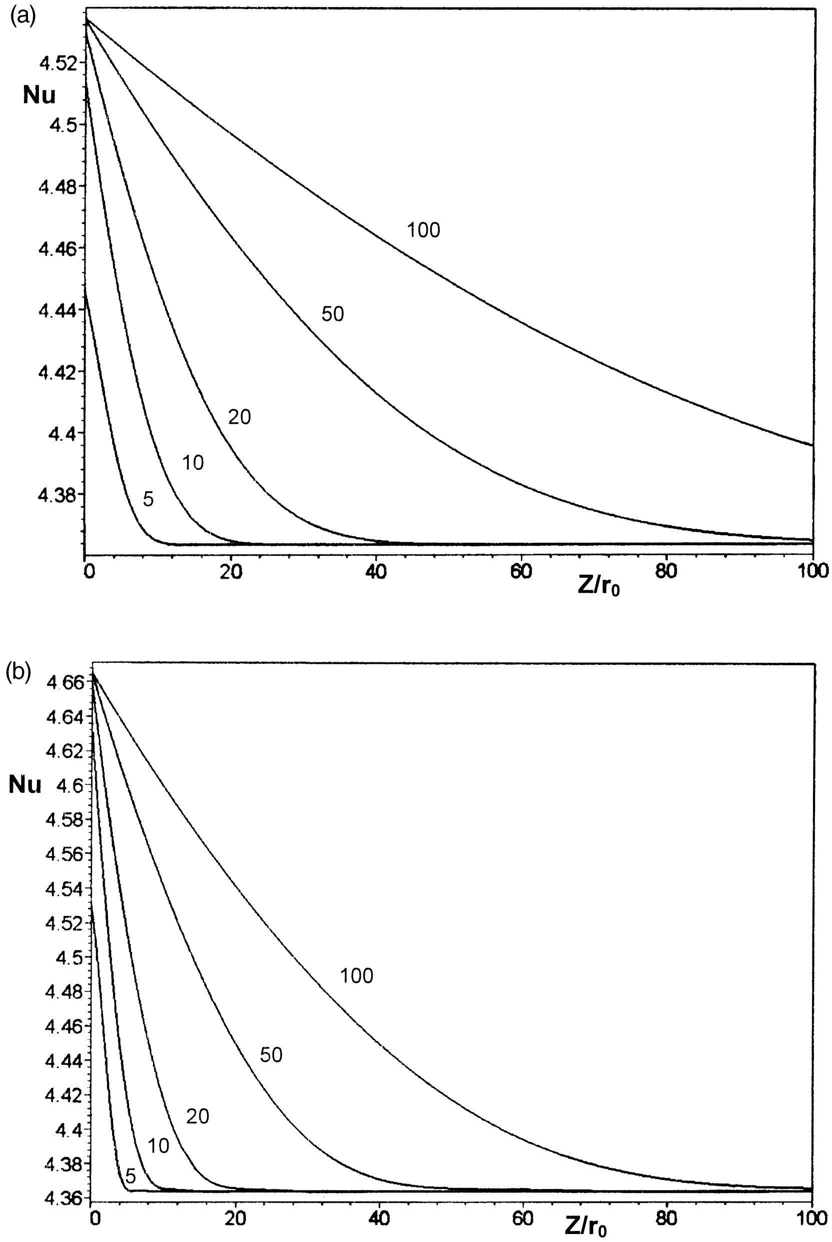

Fig. 3. Behavior of the local Nusselt number for different values of Peclet number: (a) $S=1$; (b) $S=2$.

A_j corresponding to the boundary condition of the form of Eq. (48), is the linear combination $A_j = \omega_\infty A_j^{\text{erf}} + \sum_{i=0} \omega_i A_{j,\,i}$. The same applies to the remaining constants, B–F, and consequently f_j satisfies a differential equation which is a linear combination of the differential equations for the basic solutions. The converse is obtained by the substitution of expression (48) into the differential equation for the function f_j, and observing that it reduces to the differential equations for the basic solutions when, for each successive value of i, it is made, in turns, $\omega_\infty = 1$, and $\omega_i = 0$, or $\omega_\infty = 0$, and $\omega_i = 1$, and $\omega_k = 0$ for $k \neq i$. If two functions are specified, for instance, as a consequence of the existence of two walls, then two sets of basic solutions exist, corresponding to the application of the basic set to each of the surfaces, and zero to the other.

8. Final remarks

The proposed method of solution might be compared to the approximate method due to Galerkin, where the residue of the differential equation is made orthogonal to a set of chosen functions. The proposed method does not rest on residues, and no portion of Eq. (4) is set aside, or approximated. The method is based upon the insertion of the solution into the Hilbert space of square integrable function by means of the subtraction of an asymptotic solution. This vector space admits an infinite number of complete bases, each of which can be chosen to represent any given vector. The basis chosen leads to a system of differential equations that can be exactly solved in simple fashion yielding solutions in the form of polynomials. The fact that a different set of functions was used in conjunction with the Gram–Charlier basis does not alter the quality of the solution, and is a consequence of the fact that the basis is not an orthogonal one.

The method has been tested in applications to pipe flow with parabolic velocity profile, and to the three types of boundary conditions.

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